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Poincaré maps of Duffing-type oscillators and their reduction to circle maps: I. Analytic results

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Abstract. Bifurcation diagrams and plots of Lyapunov exponents in the r–Ω-plane for Duffing-type oscillators

$$\ddot{x} + 2r\dot{x} + V'(x, \Omega t) = 0$$

exhibit a regular pattern of repeating self-similar 'tongues' with complex internal structure. We demonstrate here that this behaviour is easily understood qualitatively and quantitatively from the Poincaré map of the system in action-angle variables. This map approaches the *one-dimensional* form

$$\varphi_{n+1} = A + Ce^{-rT}\cos\varphi_n \qquad T = \pi/\Omega$$

provided e^{-rT} (but not necessarily Ce^{-rT}), r and Ω are small. We derive asymptotic (for small r, Ω) formulae for A and C for a special class of potentials V. We argue that these special cases contain all the information needed to treat the general case of potentials which obey $V'' \geqslant 0$ at all times. The essential tools of the derivation are the use of action-angle variables, the adiabatic approximation and the introduction of a non-oscillating reference solution of Duffing's equation, with respect to which the action-angle variables have to be determined. These allow the explicit construction of the Poincaré map in powers of e^{-rT}. To first order, we obtain the φ-map, which survives asymptotically. To *second* order we obtain the two-dimensional I–φ-map. In I direction it contracts by a factor e^{-rT} upon each iteration.

1. Introduction

Nonlinear oscillators and their bifurcation diagrams have been widely considered for decades, beginning with Duffing [1]. The bifurcation diagrams have a rather regular structure asymptotically, that is, for driving periods T much larger than the oscillators own characteristic time and for friction coefficients r small enough such that $\exp(-rT)$ remains distinguishable from zero. This regularity has aroused quite some interest (see, for instance, Parlitz and Lauterborn [2] and the literature cited there), but a global understanding of its mechanism has not so far been achieved.

We shall demonstrate here that this mechanism can be rather easily understood and used. It applies in principle to all equations of the type

$$\ddot{x} + 2r\dot{x} + V'(x, \Omega t) = 0 \tag{1.1}$$

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where

$$V(x,\tau) = V(x,\tau+2\pi) \tag{1.2}$$

and the following additional properties are assumed: V has only *one* extremum $x_0(\tau)$ at all times, which corresponds to a stable equilibrium of the system

$$V'(x_0(\tau),\tau) = 0 \qquad \text{and} \qquad V''(x_0(\tau),\tau) \geqslant 0. \tag{1.3}$$

In general, $V''(x_0(\tau),\tau) > 0$ but at discrete points in time, say $\tau = \tau_n$ including all periodic repetitions, we assume $V''(x_0(\tau_n),\tau_n) = 0$; i.e. the momentary harmonic frequency about the equilibrium vanishes at the times τ_n.

Now, for small r and Ω, we can apply the adiabatic theorem. In the variable $z = e^{rt}x$, the system is Hamiltonian with a slow time dependence and thus its action I stays constant most of the time. In the x-variable's phase space, $I(t)$ decays like e^{-2rt}, which is just the phase space contraction to be expected. Around the times $\Omega t_n = \tau_n$, however, the adiabatic theorem fails because the momentary harmonic frequency becomes smaller than Ω. At that point, the action, which had almost decayed to zero, gets kicked up to a new starting value $I(0)$ which depends on the angle variable $\varphi_n = \varphi(t_n)$ via $e^{-rT}\cos\varphi_n$. At the same time, φ is set essentially to zero. The increment of φ through the next adiabatic period T is obtained by integration of $\omega(I(t),\tau)$, which gives a leading term A_1T and a term $\sim e^{-rT}\cos\varphi_n$ from its I-dependence. Thus, an angular Poincaré map

$$\varphi_{n+1} = A_0 + A_1T + A(r,\Omega) + C(r,\Omega)e^{-rT}\cos\varphi_n \tag{1.4}$$

is obtained where the functions A and C are finite series of positive powers of (Ω^η/r), $\eta < 1$. Constant factors in A and C have to be determined numerically from the parameters of V. The map (1.4) has a non-trivial behaviour only in the range of parameters where $Ce^{-rT} > 1$, and there it fully explains the bifurcation diagram of the system (1.1) in the r–Ω plane. There have been previous attempts (Sato *et al* [3]) to reduce Duffing type equations to circle maps. These authors, however, did not deduce our map (1.4).

In section 2, we shall discuss the action-angle transformation for the system (1.1). Our aim is to derive the circle map and determine A and C for model systems of the type

$$\ddot{x} + 2r\dot{x} + \operatorname{sign}(x)|x|^q = |x|^\ell P(\Omega t) \tag{1.5}$$

with (essentially) arbitrary positive exponents q and ℓ. In section 3, we consider the case $\ell = 0$ and

$$P(\Omega t) = 2\Theta(\sin\Omega t) - 1 \tag{1.6}$$

i.e. a driving force which switches from $+1$ to -1 and back at intervals $T = \pi\Omega^{-1}$. Although this case in itself does not belong to the class described above, it models and correctly explains the 'kick' mechanism.

In section 4 we treat the cases

$$P(\tau) = \operatorname{sgn}(\sin\tau)|\sin\tau|^p \tag{1.7}$$

and

$$P(\tau) = |\sin \tau|^p. \tag{1.8}$$

We introduce a slowly varying *reference solution* of equation (1.5), which has an analytically accessible long time behaviour. The general solution of equation (1.5) oscillates around this reference solution; the oscillations can be treated explicitly by use of the adiabatic approximation.

Section 5 contains most of the technical details of the paper. Here the Poincaré map is derived to *second* order in e^{-rT}, and it is shown how the r–Ω-dependence of its parameters can be determined analytically. Restriction to *first* order in e^{-rT} then yields the map (1.4).

In section 6, we shall argue that the cases treated in sections 4 and 5 cover the general case, which can, if necessary, be put together from different successive maps of the type described in section 5.

2. Adiabatic approximation

We derive here the application of the adiabatic theorem to systems with friction and discuss those aspects of it which are needed in the following sections.

Consider the equation of motion

$$\ddot{x} + 2r\dot{x} + V'(x,t) = 0 \tag{2.1}$$

$$V(x,t) = \sum_{\nu=1}^{\infty} \frac{1}{\nu} a_\nu(t) x^\nu. \tag{2.2}$$

To apply the transformation

$$z = e^{rt} x \tag{2.3}$$

means that we look at the phase space trajectories derived from (2.1), which always spiral inwards, through a magnifying glass with ever increasing strength, such that we observe the Hamiltonian motion

$$\ddot{z} + W'(z,t) = 0 \tag{2.4}$$

$$\begin{aligned} W(z,t) &= e^{2rt} V(e^{-rt} z, t) - \tfrac{1}{2} r^2 z^2 \\ &= a_1(t) e^{rt} z + \tfrac{1}{2}(a_2(t) - r^2) z^2 + \tfrac{1}{3} a_3(t) e^{-rt} z^3 + \dots . \end{aligned} \tag{2.5}$$

We note that any *linear* term in V leads to an exponentially increasing term in W. This will invalidate the adiabatic approximation for large rt, which points to the (intuitively obvious) fact that one has to increase the *distance from the minimum* of V by e^{rt} to obtain a *useful* Hamiltonian description.

We shall therefore assume $a_1 = 0$ and have to take care in the following that this condition is met. *This is at the heart of our derivations.*

The harmonic term from (2.1) remains without an exponential factor in (2.5) but is supplemented by $-\frac{1}{2} r^2 z^2$. For the intended limit $r \to 0$, this addition is irrelevant,

but for computational applications with necessarily finite r it increases the accuracy if one keeps it.

The higher 'nonlinear' terms in the equation of motion (2.4) are eliminated exponentially in time, which reflects the fact that the original trajectory spirals into the harmonic region.

We now consider energies and actions with respect to z- and x-variables with τ considered to be a fixed parameter

$$F = \tfrac{1}{2}\dot{z}^2 + W(z,\tau) \tag{2.6}$$

$$E = \tfrac{1}{2}\dot{x}^2 + r x\dot{x} + V(x,\tau) = \mathrm{e}^{-2r\tau} F \tag{2.7}$$

$$J(F,\tau) = \frac{1}{2\pi}\oint \sqrt{2[F - W(\zeta,\tau)]}\,\mathrm{d}\zeta \tag{2.8}$$

$$I(E,\tau) = \frac{1}{2\pi}\oint \sqrt{2[E - \tilde{V}(\xi,\tau)]}\,\mathrm{d}\xi = \mathrm{e}^{-2r\tau} J(F,\tau) \tag{2.9}$$

$$\tilde{V}(x,\tau) = V(x,\tau) - \tfrac{1}{2}r^2 x^2. \tag{2.10}$$

Again the r-dependent terms in E and $\tilde{V}$ vanish for $r \to 0$ but ought to be kept in numerical computations. The generating function for the canonical transformation to action-angle variables (AAVs) is well known

$$S_1(z,J,\tau) = \int_{z_0}^{z} \sqrt{2[F(J,\tau) - W(\zeta,\tau)]}\,\mathrm{d}\zeta. \tag{2.11}$$

Here $F(J,\tau)$ is the inverse of $J(F,\tau)$ from (2.8); it is unique by our convexity assumption for V. We obtain

$$\varphi(z,J,\tau) = \omega \int_{z_0}^{z} \frac{\mathrm{d}\zeta}{\sqrt{F,W}} = \omega \int_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{E,\tilde{V}}} \tag{2.12}$$

where

$$\omega = \frac{\partial F(J,\tau)}{\partial J} = \frac{\partial E(I,\tau)}{\partial I} = \omega(I,\tau). \tag{2.13}$$

The adiabatic theorem guarantees that asymptotically, for r and Ω small, the motion of the system satisfies

$$I(t) = I(0)\,\mathrm{e}^{-2rt} \qquad \text{i.e.} \qquad J(t) = \text{constant} \tag{2.14}$$

and

$$\varphi(t) = \varphi(0) + \int_0^t \omega(I(\tau),\tau)\,\mathrm{d}\tau. \tag{2.15}$$

To carry out the canonical transformation induced by (2.11) it is convenient to define parameters ω_0 and a_n through the notation

$$W(z,\tau) = \omega_0^2(\tau)\left(\tfrac{1}{2}z^2 + \tfrac{1}{3}a_3(\tau)z^3 + \tfrac{1}{4}a_4(\tau)z^4 + \ldots\right). \tag{2.16}$$

We obtain the formulae

$$z(\varphi) = -R\cos\varphi + \frac{a_3}{6}R^2(\cos 2\varphi - 3) - \frac{1}{8}R^3\left[\left(\frac{a_3^2}{6} + \frac{a_4}{4}\right)\cos 3\varphi + \left(\frac{11a_3^2}{9} - \frac{3a_4}{2}\right)\cos\varphi\right] + O(R^4) \quad (2.17)$$

and (for the momentum $\dot{z}(\varphi) = p(\varphi)$)

$$\frac{1}{\omega_0}p(\varphi) = R\sin\varphi - \frac{a_3}{3}R^2\sin 2\varphi + \frac{1}{8}R^3\left[\left(\frac{a_3^2}{2} + \frac{3a_4}{4}\right)\sin 3\varphi - \left(\frac{19a_3^2}{9} - \frac{3a_4}{2}\right)\sin\varphi\right] + O(R^4) \quad (2.18)$$

Here, we have taken $R = \sqrt{2J/\omega_0}$ and defined $\varphi = 0$ by using for the lower integration limit z_0 in (2.11) the *left* turning point for each trajectory. In conjunction with these formulae for z and p we have

$$F(J,\tau) = \omega_0 J + \left(-\frac{5a_3^2}{12} + \frac{3a_4}{8}\right)J^2 + \ldots. \quad (2.19)$$

The Hamiltonian in action-angle variables—which yields the corrections to the adiabatic approximation—reads

$$\begin{aligned} H_1(J,\varphi,\tau) &= F(J,\tau) + \frac{\partial}{\partial\tau}S_1(z,J,\tau) \\ &= F(J,\tau) - \frac{1}{2}\frac{\dot{\omega}_0}{\omega_0}J\sin 2\varphi \\ &\quad + \frac{1}{2}JR\left[\left(\frac{4}{9}\frac{\dot{\omega}_0}{\omega_0}a_3 + \frac{1}{9}\dot{a}_3\right)\sin 3\varphi - \left(\frac{4}{3}\frac{\dot{\omega}_0}{\omega_0}a_3 - \dot{a}_3\right)\sin\varphi\right] + O(R^4). \end{aligned} \quad (2.20)$$

Since the angular average of $H - F$ vanishes, the true action-angle variables deviate from their adiabatic approximations (2.14) and (2.15) *on the average* only to *second* order in the derivatives $\dot{\omega}_0, \dot{a}_n$. Therefore, the correction to *first order* in $\dot{\omega}, \dot{a}_n$ for all quantities z, p, J, φ can be expressed through the adiabatic variables alone. This well known fact enhances the range of validity of our asymptotic expansion in r and Ω considerably.

We obtain for the corrected quantities φ_c and J_c

$$\varphi_c = \varphi + \frac{1}{4}\frac{\dot{\omega}_0}{\omega_0^2}\cos 2\varphi - \frac{1}{2\omega_0}R\left[\left(\frac{2}{9}\frac{\dot{\omega}_0}{\omega_0}a_3 + \frac{1}{18}\dot{a}_3\right)\cos 3\varphi - \left(2\frac{\dot{\omega}_0}{\omega_0}a_3 - \frac{3}{2}\dot{a}_3\right)\cos\varphi\right] \quad (2.21)$$

and

$$J_c = J\left\{\left(1 - \frac{1}{2}\frac{\dot{\omega}_0}{\omega_0^2}\right)\sin 2\varphi + \frac{1}{2\omega}R\left[\left(\frac{4}{9}\frac{\dot{\omega}_0}{\omega_0}a_3 + \frac{1}{9}\dot{a}_3\right)\sin 3\varphi - \left(\frac{4}{3}\frac{\dot{\omega}_0}{\omega_0}a_3 - \dot{a}_3\right)\sin\varphi\right]\right\} \quad (2.22)$$

with relative errors in second order of time derivatives and R^2 times first order of time derivatives. Here, the quantities J, φ on the RHS are the adiabatic ones from (2.14), (2.15). The quantities z and p are then obtained by inserting J_c and φ_c for J and φ in (2.17) and (2.18). We shall refer to these formulae in section 5.

The Hamiltonian (2.20) can be processed further. This is particularly useful if one intends to construct numerically Poincaré maps in AAVs for given fixed values of r and Ω instead of using the asymptotic formulae of section 5 which contain the parameter dependence in explicit analytic form.

The transformation

$$\sqrt{\frac{J}{\omega_0}}\cos\varphi = g\sqrt{\tilde{J}}\cos\tilde{\varphi} \tag{2.23}$$

$$\sqrt{J\omega_0}\sin\varphi = \frac{1}{g}\sqrt{\tilde{J}}(\sin\tilde{\varphi} - g\dot{g}\cos\tilde{\varphi}) \tag{2.24}$$

where $g(\tau)$ obeys the differential equation

$$\ddot{g} + \omega_0^2(\tau)g - \frac{1}{g^3} = 0 \tag{2.25}$$

eliminates the term linear in J from equations (2.20)–(2.22). It is generated by the function

$$S_2(\varphi, \tilde{J}, \tau) = \tilde{J}\arctan(\omega_0 g^2\tan\varphi + g\dot{g}) \tag{2.26}$$

and yields the new Hamiltonian

$$H_2(\tilde{J}, \tilde{\varphi}, \tau) = \frac{\tilde{J}}{g^2(\tau)} + O(\tilde{J}^{3/2}). \tag{2.27}$$

From this, the φ-dependence can be eliminated entirely through an ansatz for a third generating function S_3 in powers of $\tilde{J}^{1/2}$. Its coefficients are to be determined recursively from explicitly solvable linear *first* order differential equations in τ. The total effect of S_1, S_2 and S_3 could be more conveniently accomplished in one step, however, by starting directly with an ansatz for z and p of the type of equations (2.17),(2.18), where ω_0 is replaced by g^{-2} throughout.

We shall later apply the formalism to potentials $V(y,t)$ which are homogeneous functions of degree $q+1$ in y and some $x_0(t)$; more specifically

$$\begin{aligned} V(y, x_0) &= \frac{1}{q+1}[(y+x_0)^{q+1} - (q+1)yx_0^q - x_0^{(q+1)}] \\ &\quad - \frac{x_0^{(q-\ell)}}{\ell+1}[(y+x_0)^{\ell+1} - (\ell+1)yx_0^\ell - x_0^{(\ell+1)}] \\ &= \frac{q-\ell}{2}x_0^{(q-1)}y^2 + \frac{(q+\ell-1)(q-\ell)}{6}x_0^{(q-2)}y^3 + \ldots. \end{aligned} \tag{2.28}$$

Using this homogeneity, we obtain the scaling relation (dropping the $\frac{1}{2}r^2$ term from (2.10))

$$L^{q+1}E(I, x_0) = E(L^{(q+3)/2}I,\ Lx_0). \tag{2.29}$$

This yields expansions

$$E(I, x_0) = \sum_{n=1}^{\infty} c_n \, x_0^{(q+1)-\frac{n}{2}(q+3)} I^n \tag{2.30}$$

and

$$\omega(I, x_0) = c_1 x_0^{(q-1)/2} + 2c_2 x_0^{-2} I + \ldots . \tag{2.31}$$

The parameters in (2.16) are then given by

$$\omega_0^2 = (q - \ell) x_0^{q-1} \tag{2.32}$$

$$a_3 = \tfrac{1}{2}(q + \ell - 1) x_0^{-1} \tag{2.33}$$

$$a_4 = \tfrac{1}{6}[q^2 + q\ell + \ell^2 - 3(q + \ell) + 2] x_0^{-2} \tag{2.34}$$

generally we have $a_n \sim x_0^{2-n}$.

Finally

$$c_1 = (q - \ell)^{1/2} \qquad c_2 = -\tfrac{1}{48}[2q^2 + 7q\ell + 2\ell^2 - q - \ell - 1]. \tag{2.35}$$

3. Step function driving as kick mechanism

In this section we shall demonstrate, with the technically simplest model, the mechanism which leads to the map. This model turns out to correctly describe the kick mechanism encountered in section 4. We consider

$$\ddot{x} + 2r\dot{x} + x^q = \pm 1 = 2\Theta(\sin \Omega t) - 1 \tag{3.1}$$

i.e. the sign of the driving force switches at times nT. (Note that we use for convenience of notation the symbol T for the *half* period π/Ω.) We assume q to be an odd integer, to avoid the notational complication $\operatorname{sgn}(x)|x|^q$. In the final formulae, however, any $q > 1$ may be inserted. The equation of motion (3.1) is derived from a potential

$$V_{\pm}(y) = \frac{1}{q+1}(y \pm 1)^{q+1} \mp y - \frac{1}{q+1} \tag{3.2}$$

i.e. $x = x_0 + y$ with $x_0 = \pm 1$ in (2.28). The region in parameter space to be considered is given by

$$r \ll 1 \ll T. \tag{3.3}$$

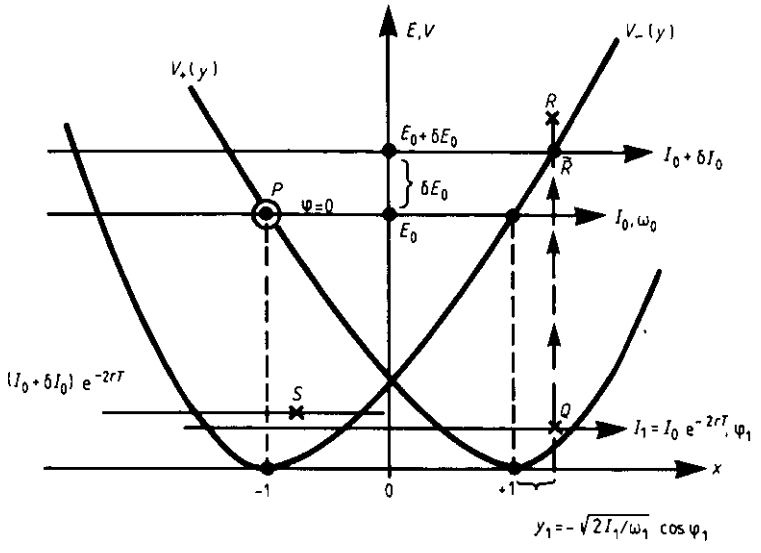

Figure 1. Step function driving as kick mechanism.

Crucial for our derivation is the further assumption $e^{-rT} \ll 1$ as we shall systematically neglect higher powers of this factor. This will be justified later.

The mechanism which leads to the map is then easily understood with the help of figure 1. Suppose, we start a motion at $t = 0$ on the right potential curve $V_+(y)$ exactly at the point P, with $E_0 = 2$, $I_0 = I(E_0)$ and $\varphi(0) = 0$. This defines E_0, I_0 and our convention for $\varphi = 0$. I_0 and ω_0 are functions of q; however, for simplicity of notation, we shall not make this explicit in our formulae.

The trajectory spirals down into the harmonic region, where it reaches some point Q at time T with $I_1 = I_0 e^{-2rT}$, and some φ_1 to which belong the quantities

$$y_1 = -\sqrt{\frac{2I_1}{\omega_1}} \cos\varphi_1 \qquad \dot{y}_1 = \sqrt{2I_1\omega_1} \sin\varphi_1 \tag{3.4}$$

proportional to e^{-rT}. The harmonic frequency here is $\omega_1 = q^{1/2}$. At that moment, the potential switches to $V_-(y)$ and the energy is instantaneously raised to point R—this is the 'kick'. The height of R above the point $\tilde{R}$ on the potential curve $V_-(y)$ is the kinetic energy (the same as the height of the point Q above the curve $V_+(y)$). It is proportional to $\dot{y}^2 \sim e^{-2rT}$ and is thus negligible to lowest order in e^{-rT}. Essential for the kick is the first order term $\delta E_0 = y_1 V_-'(2) \sim e^{-rT}$.

The next cycle of length T begins with $I(0) = I_0 + \delta I_0$ and some $\varphi(0) = \delta\varphi_0$. For this cycle, we use the notation implied by symmetry, i.e. $\varphi = 0$ at the point $\tilde{R}$.

We have (counting the angle clockwise from the point $\tilde{R}$)

$$\delta\varphi_0 = -\omega_0 \frac{\dot{y}_1}{V_-'(2)} = -\frac{\omega_0}{2}(2I_0\omega_1)^{1/2} e^{-rT} \sin\varphi_1 \tag{3.5}$$

and

$$\delta I_0 = \frac{\partial I}{\partial E}(E_0) y_1 V_-'(2) = -\frac{2}{\omega_0}\left(\frac{2I_0}{\omega_1}\right)^{1/2} e^{-rT} \cos\varphi_1. \tag{3.6}$$

To lowest order in e^{-rT} the $\delta\varphi_0$ term can be neglected compared to the angle increment $\Delta\varphi$ of (3.11); i.e. the reinjection occurs with $\varphi = 0$. ($\delta\varphi_0$ does play a role in the general case treated in section 5). This cycle ends at point S with $I_1 + \delta I_1 = e^{-2rT}(I_0 + \delta I_0)$, where δI_1 is smaller than I_1 by a factor e^{-rT} and thus again negligible. *This is the crucial step* which eliminates one phase space dimension and leads to a map in φ alone.

It means that in the full Poincaré map in I–φ coordinates the contraction towards to line $I = I_1$ is so strong that this variable may be neglected altogether. The 'kick'-mechanism is thus based on a discontinuous change of the variable y which measures the position of the oscillator with respect to the minimum of the potential; the velocity $\dot{y}$ at that moment is here irrelevant, as it only leads to contributions of higher order in e^{-rT}. This mechanism is our 'model A'. This alternation between kicks and adiabatic motion (which, as we shall see in section 4, also describes the general case) seems to lie at the base of the 'flip and twist map' described by Brown and Chua [4].

A different kick mechanism, 'model B', will also be encountered in section 4. In that model, the roles of y and $\dot{y}$ are interchanged; $\dot{y}$ is suddenly increased to $\dot{y} + 2p_0$ while y remains unchanged. We obtain expressions analogous to those of model A.

Returning to model A, we shall show below that the increment in φ during one cycle is of the form

$$\Delta\varphi = \omega_1 T + \frac{1}{r}A_2 - \frac{\omega_1 - \omega_0}{2I_0}\frac{1}{r}\delta I_0 \tag{3.7}$$

to linear order in δI_0. This, together with (3.6), yields the map

$$\varphi_{n+1} = \omega_1 T + \frac{1}{r}(A_2 + B\,e^{-rT}\cos\varphi_n) \tag{3.8}$$

where the constants A_2 and B have to be determined numerically. They contain all the relevant information on the nonlinearity of the system.

The map (3.8) can be written

$$\varphi_{n+1} = \alpha + \beta\cos\varphi_n \qquad \alpha = \omega_1 T + \frac{1}{r}A_2 \qquad \beta = \frac{B}{r}e^{-rT}. \tag{3.9}$$

It obviously yields 2π periodicity of the bifurcation diagram in the α–β parameter plane. In the r–Ω-plane, the loci of equal features (constant $\beta = Be^{-K}$) lie on the curves

$$\Omega = \pi r(K - \log r)^{-1}. \tag{3.10}$$

Ω decreases faster than r, albeit only logarithmically. Put differently, $e^{-rT} \sim r$ for the parameter range of interest; this justifies the neglect of higher orders of e^{-rT}.

We shall now complete the derivation of the relation (3.7). In

$$\Delta\varphi = \varphi(T) - \varphi(0) = \omega_1 T + \int_0^T (\omega(I) - \omega_1)\,dt \tag{3.11}$$

we use from the expansion (2.31)

$$Q(I) = \frac{\omega(I) - \omega_1}{I} = 2c_2 + 3c_3 I + \ldots \tag{3.12}$$

and $dt = -(1/2r)(dI/I)$ to obtain

$$\varphi(T) - \varphi(0) = \omega_1 T + \frac{1}{2r}\int_{I(T)}^{I(0)} Q(I)\,dI. \tag{3.13}$$

For this integration, we replace the lower bound by 0 since Q behaves regularly at 0 and $I(T)$ is negligible. We expand the upper bound to first order in δI_0 to obtain (3.8)

$$\Delta\varphi = \omega_1 T + \frac{1}{2r}\int_0^{I_0} Q(I)\,dI - \frac{\omega_1 - \omega_0}{2I_0}\frac{1}{r}\delta I_0. \tag{3.14}$$

4. The models $P(\tau) = \sin^p \tau$

We consider the system

$$\ddot{x} + 2r\dot{x} + x^q = x^\ell (\sin \Omega t)^p. \tag{4.1}$$

As we shall argue in section 6, this contains all the information required to understand the general case mentioned in section 1. We take q an odd, ℓ an even, and p an arbitrary positive integer only to avoid notational complications, our results being valid for all real values

$$q > 1 \qquad \ell = 0, 1 \text{ or } \geqslant 2 \qquad (q - \ell) > 2p > 0. \tag{4.2}$$

The necessity of the third inequality will become clear below (5.20). Arbitrary exponents are meant to imply for the driving force

model A $\quad \operatorname{sgn}(\sin \Omega t)|\sin \Omega t|^p |x|^\ell$ (4.3)

or

model B $\quad |\sin \Omega t|^p |x|^\ell$ (4.4)

and, as already mentioned, $\operatorname{sgn}(x)|x|^q$ for the anharmonic force.
For convenience, we introduce the exponent

$$\delta = \frac{p}{(q - \ell)} < \frac{1}{2}. \tag{4.5}$$

As mentioned in section 2, the adiabatic approximation requires the introduction of a new variable $y(t)$ via

$$x(t) = x_0(t) + y(t) \tag{4.6}$$

such that the total potential V for the y-motion has its minimum at $y = 0$ for all times. The 'naive' choice for $x_0(t)$, we call it $x_a(t)$ (as it is a very good approximation for most of the time), would be

$$x_a(t) = (\sin \Omega t)^\delta. \tag{4.7}$$

However, its insertion into (4.1) via (4.6) yields, of course, additional terms from the derivatives of x_a which, though of higher order in Ω, diverge for $t \to 0$ and thus spoil the desired property of V.

The equation of motion in $y(t)$ reads

$$\ddot{y} + 2r\dot{y} + (y + x_0)^q - x_0^q - [(y + x_0)^\ell - x_0^\ell] \sin^p \Omega t = -[\ddot{x}_0 + 2r\dot{x}_0 + x_0^q - x_0^\ell \sin^p \Omega t]. \tag{4.8}$$

We require the RHS to vanish, thus x_0 itself must be a solution of (4.1), the 'reference solution'. For the application of the adiabatic approximation to the motion described by the LHS of (4.8), $x_0(t)$ must be a *special* solution, namely a creeping solution which only varies on some time scale $\Omega^{-\eta} \gg 1$, and *not* on time scale 1, as the general solution does, i.e. *it must not oscillate.*

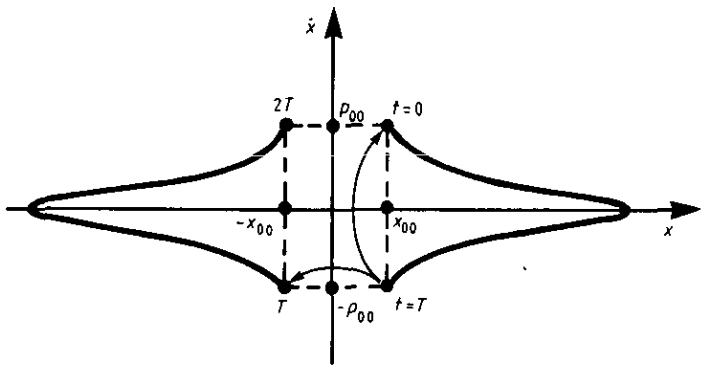

Figure 2. Reference solution with jumps.

Such a solution indeed exists during one half period T. It can be defined through an asymptotic expansion in r and Ω. It will be close to x_a except near $t = 0$ and $t = T$. Its qualitative phase portrait (and the symmetric one for the next half period of model A) is shown in figure 2. These reference solutions *do not join together* at $t = T$, as they do in the harmonic case. Instead, for model A, it jumps from x_{00} to $-x_{00}$ with fixed $\dot{x}_0$, whereas for model B, x_0 remains unaltered and $\dot{x}_0$ jumps from $-p_{00}$ to $+p_{00}$. *This provides exactly the two kick mechanisms discussed in the previous section!*

During each half period T, the potential $V(y, x_0(t), t)$ varies slowly in time and thus allows for the adiabatic approximation. At $t = nT$, the action I is always kicked up again.

The determination of the kick parameters requires further investigation on $x_0(t)$. This we shall consider next.

To investigate $x_0(t)$ in the vicinity of $t = 0$, it suffices to consider the equation of motion

$$\ddot{x} + 2r\dot{x} + x^q = x^\ell (\Omega t)^p. \tag{4.9}$$

We rescale the variables as

$$x(t) = \Omega^\gamma \xi(\tau) \qquad t = \Omega^{-\eta} \tau \tag{4.10}$$

with the exponents

$$\gamma = \frac{2\delta}{2+(q-1)\delta} \qquad \delta = \frac{p}{q-\ell} \qquad \eta = \frac{(q-1)\delta}{2+(q-1)\delta} \tag{4.11}$$

to obtain

$$\ddot{\xi} + 2\rho\dot{\xi} + \xi^q = \xi^\ell \tau^p \tag{4.12}$$

$$\rho = r\Omega^{-\eta}. \tag{4.13}$$

The smooth solution $\xi_0(\tau)$ of (4.12), which then corresponds to the desired function $x_0(t)$ near $t = 0$, can be approximated by an asymptotic series, which is obtained by iteration of

$$\xi_{n+1}(\tau) = [\tau^p - \xi_n^{-\ell}(\ddot{\xi}_n + 2\rho\dot{\xi}_n)]^{1/(q-\ell)} \tag{4.14}$$

starting with an initial function $\xi_1(\tau) = \tau^\delta$. The series has the form

$$\xi_0(\tau) = \tau^\delta\left(1 + \sum_{n=1}^{\infty} P_n(\rho\tau)\tau^{-n\lambda}\right) \qquad \lambda = 2 + (q-1)\delta \tag{4.15}$$

where the P_n are certain polynomials of order n. It definitely diverges as $P_n(0)$ grows roughly like $(n!)^2$. Nevertheless, it is an asymptotic approximation valid for large τ and serves several important purposes:

(*a*) it yields reliable initial conditions at large τ for the determination of $\xi_0(\tau)$ by numerical integration of the equation of motion (4.12).

(*b*) it shows that asymptotically, for small ρ, the ρ-dependence of the required solution is negligible. From (4.13) and (3.10), we have $\rho \approx \Omega^{1-\eta} \ll 1$. *The reference solution $x_0(t)$ can thus be determined with $r = 0$!* This can be made intuitive in the following manner (figure 3).

The minimum of the total potential in (4.1) *decelerates* during the first quarter period. In order for $x_0(t)$ to follow this deceleration without oscillations, the motion must start with a very particular initial velocity towards the right and a very particular position on the RHS of the minimum (see figure 3). During some initial time interval of order $\Omega^{-\eta}$, the required relative deceleration $\ddot{x}/\dot{x}$ is much larger than the friction coefficient r. On the other hand, at the time $\tau \approx 1/r$, when the friction effect is being felt, $x_0(t)$ has already approached $x_a(t)$ very closely and the velocity is small.

The same consideration also demonstrates that the trajectories $x_0(t)$ of figure 2 remain in their respective half planes, i.e. they do not cross the $\dot{x}$-axis.

(*c*) The solution (4.15) shows that $\xi_0(\tau)$ approaches τ^δ algebraically. Described in the original variables, this happens on a time scale $\Omega^{-\eta}$. Thus, we have fully separated time scales asymptotically for $\Omega \to 0$: $x_0(t)$ approaches $(\Omega t)^\delta$ on the scale $\Omega^{-\eta}$ whereas the difference between Ωt and $\sin \Omega t$ is only being felt on the scale $\Omega^{-1} \gg \Omega^{-\eta} \gg 1$.

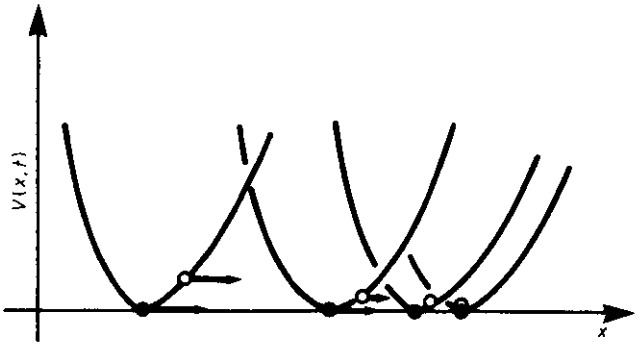

Figure 3. Motion of the potential and of the reference solution.

We arrived at the following description for $x_0(t)$ in the upper right quarter of the phase plane of figure 2 (the other quarters are obtained by symmetry)

$$x_0(t) = (\sin \Omega t)^\delta + \Delta x_0(t) \tag{4.16}$$

where

$$\Delta x_0(t) = \Omega^\gamma(\xi_0(\tau) - \tau^\delta) \qquad \tau = \Omega^\eta t \tag{4.17}$$

and $\xi_0(\tau)$ is the unique slowly varying solution of

$$\ddot{\xi}_0 + \xi_0^q = \xi_0^\ell \tau^p. \tag{4.18}$$

This reference solution is similar to and is related to asymptotic solutions of Duffing's equations considered by Byatt-Smith [1].

In particular, we define

$$\xi_1 = \xi_0(0) \qquad \xi_2 = \dot{\xi}_0(0). \tag{4.19}$$

ξ_1 and ξ_2, like I_0 and ω_0 from section 3, belong to a set of about a dozen 'universal' numbers, that have to be determined numerically to obtain all the prefactors of the map. By 'universal' we mean independent of r and Ω. They do depend on the exponents ℓ, p and q and on whether one considers model A or B.

To numerically calculate creeping solutions for finite values of Ω and r one generates initial conditions for the differential equation at $t = T/2$ by substituting the series

$$x_0(t) = \sin^\delta \Omega t(1 + \sum_{n=1}^{\infty} R_n \ \Omega^{-2n} \sin^{-n\lambda} \Omega t) \tag{4.20}$$

into the equation (4.1). The coefficients R_n depend on $\cos \Omega t$, $\sin \Omega t$ and r/Ω. The series is asymptotic for $\Omega \to 0$ and yields useful initial conditions at $t = T/2$ for creeping solutions if $\Omega^2 \lesssim 10^{-1}$.

5. Determination of the map parameters

In this section we shall derive the angular Poincaré map

$$\varphi_{n+1} = \alpha + \beta \cos \varphi_n \tag{5.1}$$

for systems of type (4.1) and give expressions for its parameters α and β in terms of r and Ω, valid asymptotically for small r and Ω. We shall obtain representations

$$\alpha = A_0 + A_1 T + \sum_{\nu=2}^{N} A_\nu F_\nu(r, \Omega) \tag{5.2}$$

$$\beta = \mathrm{e}^{-rT}\left[C + B_1 \sum_{\nu=2}^{N} (\nu - 1) A_\nu F_\nu(r, \Omega)\right]. \tag{5.3}$$

Here, the constants A_ν, B_1 and C will be expressed in terms of the basic set of numbers mentioned earlier. The functions F_ν are proportional to positive powers of Ω^η / r when $\Omega \ll r$, but we need not make this additional assumption. The formulae (5.2), (5.3) are asymptotic in the sense that all terms which are smaller by factors e^{-rT} or $(\Omega^\eta / r)^{-c}$, compared to those kept, are systematically neglected. In the following we shall derive our results by first crudely tracing the generation of the map and then filling in details later.

Suppose, for the moment, that the adiabatic approximation for the motion with respect to the reference solutions were rigorously valid at all times. At the end of the $(n-1)$th cycle the system coordinates are close to the endpoint of the reference solution, denoted by $2T$ on the LHS of figure 2.

The nth cycle from $t = 0$ to $t = T$ then starts with initial AAVs, which are conveniently written as

$$I_n(0) = \Omega^{2\gamma+\eta}(I_a + \delta I_n) = \Omega^{2\gamma+\eta} I_a (1 + \mathrm{e}^{-rT} i_n) \tag{5.4}$$

$$\varphi_n(0) = \varphi_a + \delta\varphi_n. \tag{5.5}$$

Here, we have denoted by I_a and φ_a those values which are obtained if we start *exactly* at the endpoint $2T$ of the LHS reference solution in figure 2. The scaling factor $\Omega^{2\gamma+\eta}$ transforms I into the ξ,τ scale of (4.10) such that I_a becomes asymptotically independent of Ω. Continuing to $t = T$, I_n and φ_n evolve into

$$I_n = I_n(T) = \mathrm{e}^{-2rT} I_n(0) \qquad \varphi_n = \varphi_n(T) \tag{5.6}$$

where the φ_n are derived below. Near T the motion is close to harmonic as in the model of section 3; its coordinates and velocities are then given by linear combinations of the quantities

$$u_1 = \mathrm{e}^{-rT}(1 + \mathrm{e}^{-rT} i_n)^{1/2} \mathrm{e}^{\mathrm{i}\varphi_n} \qquad \text{and} \qquad u_2 = u_1^* \tag{5.7}$$

with factors $(I_a / 2\omega_0)^{1/2}$ and scaling powers of Ω, the latter depending on whether one uses the x, t or the ξ, τ scaling. Performing a specular reflection in the $x, \dot{x}$ plane (considering model A), the system is again near the point $2T$ of figure 2. More

specifically, it is apparent from the analogous discussion in section 3 that its deviation from the endpoint of the reference solution is of order e^{-rT}. The initial conditions for the $(n+1)$th cycle can now be expanded in powers of the small quantities u_ν and we can write to *second* order (using the usual summation convention)

$$e^{-rT} i_{n+1} = K_\nu u_\nu + \tfrac{1}{2} K_{\nu\mu} u_\nu u_\mu \tag{5.8}$$

$$\delta\varphi_{n+1} = P_\nu u_\nu + \tfrac{1}{2} P_{\nu\mu} u_\nu u_\mu . \tag{5.9}$$

Since I and φ are real, these coefficients are more conveniently expressed by amplitudes and phases as

$$K_1 = K_2^* = \tfrac{1}{2} B_1 e^{i\chi_1} \qquad P_1 = P_2^* = \tfrac{1}{2} C_1 e^{i\psi_1} \tag{5.10}$$

$$K_{11} = K_{22}^* = \tfrac{1}{2} B_2 e^{i\chi_2} \qquad P_{11} = P_{22}^* = \tfrac{1}{2} C_2 e^{i\psi_2} \tag{5.11}$$

$$K_{12} = K_{21}^* = \tfrac{1}{2} B_3 \qquad P_{12} = P_{21}^* = \tfrac{1}{2} C_3 \quad \text{real.} \tag{5.12}$$

Also, the transformation from u_ν to I, φ is canonical; this requires

$$2\,\mathrm{Im}(K_1 P_2) = \tfrac{1}{2} B_1 C_1 \sin(\chi_1 - \psi_1) = 1 \tag{5.13}$$

and

$$K_{11} P_2 + K_1 P_{12} = P_{11} K_2 + P_1 K_{12}. \tag{5.14}$$

Together with I_a and φ_a we have thus twelve numerical constants which determine the map to second order in e^{-rT}. We shall show below that these constants are asymptotically independent of Ω and r and thus 'universal'; they do depend on p, q and ℓ, of course.

The map is thus a systematic expansion in powers of e^{-rT} and to second order in e^{-rT} we obtain for the action

$$i_{n+1} = (1 + \tfrac{1}{2} e^{-rT} i_n) B_1 \cos(\varphi_n + \chi_1) + e^{-rT}[B_2 \cos(2\varphi_n + \chi_2) + B_3] \tag{5.15}$$

and for the angle

$$\begin{aligned}\varphi_{n+1}(0) = \varphi_a &+ e^{-rT}(1 + \tfrac{1}{2} e^{-rT} i_n) C_1 \cos(\varphi_n + \psi_1) \\ &+ e^{-2rT}(C_2 \cos(2\varphi_n + \psi_2) + C_3).\end{aligned} \tag{5.16}$$

To complete the map, we need the evolution of $\varphi_{n+1}(0)$ to $\varphi_{n+1}(T)$. We have

$$\varphi_{n+1}(T) = \varphi_{n+1}(0) + \int_0^T \omega_a(I_{n+1}(t), t)\,dt. \tag{5.17}$$

Except for t values close to 0 and T, the function $x_0(t)$ is well approximated by $x_a(t) = (\sin \Omega t)^\delta$. Consequently, the expansion (2.31) is valid with $x_0 = x_a(t)$. We write

$$\omega_a(I(t), t) = \sum_{\nu=1}^{\infty} \nu c_\nu I^{\nu-1}(t) (\sin \Omega t)^{-2\delta_\nu} \tag{5.18}$$

where we introduce

$$\delta_\nu = \frac{\delta}{2}\left[\frac{\nu}{2}(q+3)-(q+1)\right] \qquad \kappa_\nu = 1-2\delta_\nu. \tag{5.19}$$

We rewrite this using $\gamma = \delta(1-\eta)$

$$\omega_a = c_1 \sin^{-2\delta_1}(\Omega t) + \sum_{\nu\geqslant 2} \nu c_\nu (I_a + \delta I)^{\nu-1} \Omega^{\eta\kappa_\nu} \left(\frac{\sin \Omega t}{\Omega}\right)^{-2\delta_\nu} e^{-2(\nu-1)\tau t}. \tag{5.20}$$

Consider this expression in the extreme asymptotic limit $1 \gg r \gg \Omega$ where we may linearize $\sin \Omega t$ in the region where $e^{-2\tau t}$ is non-negligible. The νth term in the sum (5.20) contributes a term to the angle increment $\int \omega_a \, dt$ proportional to

$$F_{\nu,\text{asymptotic}} = \Omega^{\eta\kappa_\nu} \int_0^\infty \tau^{-2\delta_\nu} e^{-2(\nu-1)r\tau} \, d\tau = \frac{\Gamma(\kappa_\nu)}{(\nu-1)^{\kappa_\nu}} \left(\frac{\Omega^\eta}{2r}\right)^{\kappa_\nu}. \tag{5.21}$$

We are interested only in terms with *positive* κ_ν; any term with negative κ_ν yielded an asymptotically vanishing contribution from the upper integration limit and diverged at $\tau = 0$. Consequently, we sum the terms in (5.18) only up to $\kappa_{\max} = N$, such that $\kappa_N > 0$, $\kappa_{N+1} < 0$. (We do not consider the rather special case that some $\kappa_N = 0$).

The condition $\delta < \frac{1}{2}$ of (4.4) guarantees $N \geqslant 2$, otherwise no I-dependence would survive in (5.20) and thus β-values of interest could not be realized with $r \ll 1$, $\Omega \ll 1$ *and* $e^{-rT} \ll 1$ simultaneously.

For *finite* r and Ω we obtain from the adiabatic approximation

$$\varphi_{n+1}(T) - \varphi_{n+1}(0) = \int_0^T \omega_a \, dt = A_1 T + \sum_{\nu=2}^{N} \nu c_\nu (I_a + \delta I)^{\nu-1} F_\nu(r, \Omega) \tag{5.22}$$

with

$$A_1 = \frac{c_1}{\sqrt{\pi}} \frac{\Gamma(\frac{\kappa_1}{2})}{\Gamma(\frac{\kappa_1+1}{2})} \tag{5.23}$$

and

$$F_\nu(r, \Omega) = \Omega^{(\eta-1)\kappa_\nu} F\left(\delta_\nu, (\nu-1)\frac{r}{\Omega}\right) \tag{5.24}$$

where F is given by

$$\begin{aligned} F(\delta, \mu) &= \int_0^\pi e^{-2\mu t} (\sin t)^{-2\delta} \, dt \\ &= 2^{-\kappa} e^{-\pi\mu} \frac{2\pi\Gamma(\kappa)}{\Gamma(1-\delta+i\mu)\Gamma(1-\delta-i\mu)}. \end{aligned} \tag{5.25}$$

We could not find this integral in tables and arrived at equation (5.25) through analytical continuation to $\beta = e^{-i\pi/2}$ of [5]

$$\int_0^\infty e^{-2\mu t} \left(\frac{1}{\beta} \sinh \beta t\right)^{-2\delta} dt = (2\beta)^{-\kappa} \Gamma(\kappa) \frac{\Gamma(\delta + \mu/\beta)}{\Gamma(1-\delta+\mu/\beta)}. \tag{5.26}$$

Collecting the terms from equations (5.5), (5.9), (5.15), (5.16) and (5.20) we obtain for the φ-map—to second order in e^{-rT}

$$\begin{aligned}\varphi_{n+1} &= \varphi_{n+1}(0) + \int_0^T \omega_a(I_{n+1}(t), t)\,dt \\ &= \varphi_a + (1 + \tfrac{1}{2}e^{-rT} i_n) C_1 \cos(\varphi_n + \psi_1) + e^{-rT}[C_2 \cos(2\varphi_n + \psi_2) + C_3] \\ &\quad + A_1 T + \sum_{\nu=1}^{N} A_\nu F_\nu(r, \Omega)(1 + e^{-rT} i_{n+1})^{\nu-1} \end{aligned} \tag{5.27}$$

with

$$A_\nu = \nu c_\nu I_a^{\nu-1} \qquad (\nu \geqslant 2). \tag{5.28}$$

Sorting the terms of φ_{n+1} in increasing powers of e^{-rT}, we obtain

$$\varphi_{n+1} = O_0 + e^{-rT} O_1 + e^{-2rT} O_2 \tag{5.29}$$

where

$$O_0 = \varphi_a + A_1 T + \sum_{\nu=2}^{N} A_\nu F_\nu(r, \Omega) \tag{5.30}$$

$$O_1 = B_1 \sum_{\nu=2}^{N} (\nu - 1) A_\nu F_\nu(r, \Omega) \cos(\varphi_n + \chi_1) + C_1 \cos(\varphi_n + \psi_1) \tag{5.31}$$

and

$$\begin{aligned} O_2 = \sum_{\nu=2}^{N} (\nu - 1) A_\nu F_\nu(r, \Omega) &[B_2 \cos(2\varphi_n + \chi_2) + B_3 \\ &+ \tfrac{1}{2}(\nu - 2) B_1^2 \cos^2(\varphi_n + \chi_1) + \tfrac{1}{2} B_1 i_n \cos(\varphi_n + \chi_1)] \\ &+ \tfrac{1}{2} C_1 i_n \cos(\varphi_n + \psi_1) + C_2 \cos(2\varphi_n + \psi_2) + C_3. \end{aligned} \tag{5.32}$$

These formulae contain all terms to second order in e^{-rT} and to all non-negative (fractional) powers of (Ω^η / r).

In the derivation of the map, equations (5.15) and (5.29)–(5.32), we used two unrealistic simplifications:

(a) The equation of motion of the y variable from equation (4.8)

$$\ddot{y} + 2r\dot{y} + (y + x_0)^q - x_0^q - [(y + x_0)^\ell - x_0^\ell] \sin^p \Omega t = 0 \tag{5.33}$$

cannot be handled by the adiabatic approximation near $t = 0$ and $t = T$ because there $\dot{x}_0 / x_0$ is *not* small (see figure 2).

(b) In the same regions, $(\sin \Omega t)^6$ *does not* yield a reasonable approximation to $x_0(t)$, therefore (5.23) is not correct in this region.

Nevertheless, the correction for both effects is completely absorbed into a proper choice of the constants I_a, φ_a, K and P; *the map equations remain unchanged.*

We shall describe here the principal idea only; technical details will be given in a later paper.

In the vicinity of $\tau = 0$, we use the scaling of section 4 and solve

$$\ddot{\xi} + \xi^q - \xi^\ell \tau^p = 0 \tag{5.34}$$

as a good approximation for the motion within the asymptotically interesting range of parameters τ, Ω. Actually, this 'vicinity' becomes arbitrarily large in the asymptotic limit and we may consider initial and final times $-\tau_- \gg 1$ and $\tau_+ \gg 1$ large enough that at these times the motion of $\xi(\tau) - \xi_0(\tau)$ has become harmonic and adiabatic.

For initial conditions at a *large negative* time τ_- (observe the specular reflection—we consider model A), we use

$$\xi(\tau_-) = -\xi_0(-\tau_-) + \left(\frac{I_a}{2\omega_0(\tau_-)}\right)^{1/2} (u_1 e^{i\Delta\varphi_-} + u_2 e^{-i\Delta\varphi_-}) \tag{5.35}$$

$$\dot{\xi}(\tau_-) = \dot{\xi}_0(-\tau_-) + i\left(\frac{\omega_0(\tau_-) I_a}{2}\right)^{1/2} (u_1 e^{i\Delta\varphi_-} - u_2 e^{-i\Delta\varphi_-}). \tag{5.36}$$

Here

$$\omega_0^2(\tau) = (q - \ell)|\tau|^{(q-1)\delta} \tag{5.37}$$

and $\Delta\varphi_-(\tau_-)$ will be defined below. At large positive times τ_+ the solution has the form

$$\xi(\tau_+) = \xi_0(\tau_+) - \left(\frac{2I_+}{\omega_0(\tau_+)}\right)^{1/2} \cos(\varphi_a + \Delta\varphi_+(\tau_+)) \tag{5.38}$$

$$\dot{\xi}(\tau_+) = \dot{\xi}_0(\tau_+) + (2\omega_0(\tau_+) I_+)^{1/2} \sin(\varphi_a + \Delta\varphi_+(\tau_+)). \tag{5.39}$$

For $u_1 = u_2 = 0$, we put

$$I_+ = I_a \qquad \text{and} \qquad \Delta\varphi_+ = \int_0^{\tau_+} \omega_a(I_a, \tau)\, d\tau \tag{5.40}$$

thus defining I_a and φ_a. We may now insert I_a into the initial conditions (5.35) and (5.36) and put

$$I_- = I_a |u_1|^2 \qquad \text{and} \qquad \Delta\varphi_- = -\int_0^{|\tau_-|} \omega_a(I_-, \tau)\, d\tau. \tag{5.41}$$

For sufficiently small $u_1 = u_2^*$ we obtain then at τ_+

$$I_+ = I_a + \delta I \qquad \text{and} \qquad \Delta\varphi_+ = \delta\varphi + \int_0^{\tau_+} \omega_a(I_+, \tau)\, d\tau \tag{5.42}$$

with quantities δI and $\delta\varphi$ which are (asymptotically in τ_+) independent of τ_+ and which have the expansion (5.8) and (5.9) in powers of u_1 and u_2.

This construction ensures that all effects of the transition through the non-adiabatic and non-harmonic region near $\tau = 0$ as well as the effect of $\omega - \omega_a \neq 0$ in this region are fully absorbed in the well defined limiting quantities I_a, φ_a, δI, $\delta\varphi$ and the expansion coefficients of the latter in powers of u_ν. These quantities are accessible only numerically.

This completes the derivation of the map.

6. Summary and conclusions

We have shown, that the bifurcation behaviour of the generalized Duffing equation

$$\ddot{x} + 2r\dot{x} + x^q = x^\ell \sin^p \Omega t \tag{6.1}$$

can be understood asymptotically from a *one dimensional* (angle-) Poincaré map

$$\varphi_{n+1} = \alpha + \beta \cos \varphi_n . \tag{6.2}$$

Two aspects of the 'asymptotic' limit are *essential* assumptions with physical contents; a third one is more technical.

Firstly, we assumed the validity of the adiabatic approximation which is sufficiently well fulfilled if r and Ω are of order 10^{-1} or less. Secondly, we argued that the Poincaré map for one half cycle is essentially one dimensional and only depends on the angle φ if e^{-rT} is small enough; $rT \gtrsim 2$ suffices. The obvious reason for the latter being, that the action contracts by e^{-2rT} during each half cycle. Both conditions are indispensible in obtaining the map (6.2), although we believe that qualitatively the structures of the 'tongues' in the Duffing's equations' bifurcation diagram remain the same even outside this asymptotic region.

Thirdly, we have derived analytic expressions for the constants α and β in terms of r, Ω and numerical constants, which are summarized below. For these expressions to be nearly correct, one needs to go further into asymptopia. The factor e^{-rT} roughly estimates the *relative* error in the $\beta \cos \varphi_n$ term of (6.2); including a deviation from the pure cosine behaviour as one sees in (5.31). In addition, the quantity Ω^η / r must be rather small to justify the neglect of higher terms in the ν sums.

Thus there exists a wide region in r–Ω parameter space, where a one-dimensional φ-map with first and second Fourier components is a very good approximation to the Poincaré map for (6.1), but where analytic expressions for the coefficients are not easily available. Actually, for applications, this is the more interesting region. Although following section 5 one may derive corrections to the formulae given below, it may be more practical to simply determine a one dimensional map by numerical integration of (6.1) for one half period T. This will be described in the forthcoming paper. The asymptotic AAVs are best initialized and compared with map values at the point $t = T/2$.

Collecting all the bits and pieces from the previous sections, we have obtained the following asymptotic expressions for the parameters of the map (6.2)

$$\alpha = A_0 + A_1 T + \sum_{\nu=2}^{N} A_\nu F_\nu(r, \Omega) \tag{6.3}$$

$$\beta = e^{-rT} \left[C + B_1 \sum_{\nu=2}^{N} (\nu - 1) A_\nu F_\nu(r, \Omega) \right] \tag{6.4}$$

where $A_0 = \varphi_a + \chi_1$, B_1 and $C = C_1 \cos(\psi_1 - \chi_1)$ are related to the numerical constants defined in section 5. We have shifted all angles by χ_1 to obtain the pure cosine behaviour in (6.2). The omitted term $C_1 \sin(\psi_1 - \chi_1) \sin \varphi_n$ contributes a change in amplitude and a phase shift which are of order $(\Omega^\eta / r)^{-\kappa_2}$, but neglecting this term is valid only in the extreme asymptotic limit.

The factors are

$$A_1 = \frac{c_1}{\sqrt{\pi}} \frac{\Gamma(\frac{\kappa_1}{2})}{\Gamma(\frac{\kappa_1+1}{2})} \qquad A_\nu = \nu c_\nu I_a^{\nu-1} \tag{6.5}$$

$$F_\nu = 2^{-\kappa_\nu} \Omega^{(\eta-1)\kappa_\nu} e^{-\pi(\nu-1)r/\Omega} \frac{2\pi\Gamma(\kappa_\nu)}{\Gamma(1-\delta_\nu+i\frac{(\nu-1)r}{\Omega})\Gamma(1-\delta_\nu-i\frac{(\nu-1)r}{\Omega})}$$

$$\approx \frac{\Gamma(\kappa_\nu)}{(\nu-1)^{\kappa_\nu}} \left(\frac{\Omega^\eta}{2r}\right)^{\kappa_\nu} \tag{6.6}$$

which contain the fourth numerical constant I_a and the Taylor coefficients nc_n from (2.34).

The exponents were

$$\delta = \frac{p}{q-\ell} < \frac{1}{2} \qquad \eta = \frac{(q-1)\delta}{2+(q-1)\delta} \tag{6.7}$$

$$\delta_\nu = \frac{\delta}{2}\left[\frac{\nu}{2}(q+3) - (q+1)\right] \qquad \kappa_\nu = 1 - 2\delta_\nu \tag{6.8}$$

and N is to be determined by $\kappa_N > 0, \kappa_{N+1} < 0$.

The foregoing considerations are in principle easily extended to the general case of a potential $V(x, \Omega t)$ as described in section 1. In the asymptotic region of small Ω and r, most of the time the system oscillates harmonically around the slowly moving minimum of V, given by Equation (1.3). This motion determines the leading term of the angle increment as $A_1 T$ with

$$A_1 = \frac{1}{\pi} \int_{s_n}^{s_n+1} [V''(x_0(\tau), \tau)]^{1/2} \, d\tau \tag{6.9}$$

where $s_n = \Omega t_n$ and $s_{n+1} = \Omega t_{n+1}$ are two subsequent time instants at which $V''(x_0(s), s) = 0$. They span a smooth 'period' between two kicks. Everything else happens in a narrow region around the t_n, where V may be approximated by its lowest order terms in x and t. For simplicity of notation, let us take $t_n = 0$ and $x_0(t_n) = 0$. Then, $V'(0,0) = 0$, $V''(0,0) = 0$, and we may write, to lowest order in x and t, with proper scaling of time and length

$$V(x, \Omega t) \sim \frac{1}{q+1}|\xi|^{q+1} - \frac{1}{\ell+1}\mathrm{sgn}(\xi)|\xi|^{\ell+1}\sigma_\tau|\tau|^p \tag{6.10}$$

i.e.

$$V'(x, \Omega t) \sim \mathrm{sgn}(\xi)|\xi|^q - \sigma_\tau|\xi|^\ell]\tau|^p. \tag{6.11}$$

One has to assume $q > 1$ and $q > \ell$ lest the assumptions on V'' and on x_0, respectively, are violated. (The condition $2p < (q - \ell)$ is *not necessarily* fulfilled. In that case, our theory is not applicable!) The sign function $\sigma_\tau = (\tau/|\tau|)^\nu$, $\nu = 1$ or 0, allows for the two models A or B respectively for which $x_0(t)$ crosses or touches the t-axis. Except for additional numerical factors

which enter through the foregoing scaling, we obtain the map (6.2) from $\varphi(t_n)$ to $\varphi(t_{n+1})$, where the factors now have to be determined by considering the motion through two previous 'periods'. Thus, if several zeros at different time instances occur in V'' during one period $2\pi/\Omega$, one had to introduce one map with possibly different parameters for each t_n. The bifurcation diagram is then determined by the finite sequence of maps which describes the alternation between adiabatic motion and kicks which the system experiences during one full cycle. In the models of section 4, we exploited the symmetry between both half periods to reduce all effects to one single map (6.2).

The original Duffing equation

$$\ddot{X} + 2R\dot{X} + X^3 + X = A\sin\omega\tau \tag{6.12}$$

does not obviously belong to the class of nonlinear systems we have considered since $V'' > 0$ throughout in (6.12). Its bifurcation diagram is often considered [2] in the A–ω plane for fixed R and large A and has features very similar to the one without the linear restoring force. This is explained as follows.

Let $a = A^{1/3} > 0$ and rescale

$$x = \frac{X}{a} \qquad t = a\tau \qquad \Omega = \frac{\omega}{a} \qquad r = \frac{R}{a} \tag{6.13}$$

to obtain

$$\ddot{x} + 2r\dot{x} + x^3 + (r/R)^2 x = \sin\Omega t. \tag{6.14}$$

The asymptotic behaviour in the r–Ω parameter plane with R held fixed is now described by our theory with $q = 3$, $\ell = 0$, $p = 1$. The term $(r/R)^2 x$ is negligible if $(r\Omega^{-\gamma}) \ll R$; we had neglected a similar term in section 2.

An interesting question arises how to understand bifurcation diagrams for systems, the potentials of which do not obey all assumptions made in section 1.

One could consider systems where $V'' > 0$ remains finite asymptotically in r and Ω for all times t. Our adiabatic approximation scheme breaks down in this case as it yields $I(t) \sim \mathrm{e}^{-2rt} I(0)$ forever. It is conceivable that such systems have no bifurcations at all in the asymptotic limit. Otherwise, one expects that $r/\Omega \lesssim 1$ is the parameter regime of interest. One could imagine, then, that some correction to the adiabatic approximation leads to an equation for slow motion of I, which might yield a map upon completion of a cycle.

Another possibility for V is that $V'' = 0$ does occur, but that V has a multiple well structure, at least during some time intervals (for instance, the periodically driven Josephson junction, a system of great interest). In that case, the adiabatic approximation applies during most of the time, but switching between different wells will yield a wealth of new phenomena, which are not likely to be amenable to the simple type of analysis we have used here.

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